

Quintessence: A Review

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Abstract.

Quintessence is a canonical scalar field introduced to explain the late-time cosmic acceleration. The cosmological dynamics of quintessence is reviewed, paying particular attention to the evolution of the dark energy equation of state w . For the field potentials having tracking and thawing properties, the evolution of w can be known analytically in terms of a few model parameters. Using the analytic expression of w , we constrain quintessence models from the observations of supernovae type Ia, cosmic microwave background, and baryon acoustic oscillations. The tracking freezing models are hardly distinguishable from the Λ -Cold-Dark-Matter (Λ CDM) model, whereas in thawing models the today's field equation of state is constrained to be $w_0 < -0.7$ (95 % CL). We also derive an analytic formula for the growth rate of matter density perturbations in dynamical dark energy models, which allows a possibility to put further bounds on w from the measurement of redshift-space distortions in the galaxy power spectrum. Finally we review particle physics models of quintessence—such as those motivated by supersymmetric theories. The field potentials of thawing models based on a pseudo-Nambu-Goldstone boson or on extended supergravity theories have a nice property that a tiny mass of quintessence can be protected against radiative corrections.

1. Introduction

The observational discovery of the late-time cosmic acceleration from the Supernovae type Ia (SN Ia) opened up a new research area in modern cosmology [1, 2]. About 70 % of the energy density of the Universe today consists of an unknown component called dark energy. This has been also confirmed by other observations— such as Cosmic Microwave Background (CMB) [3, 4] and Baryon Acoustic Oscillations (BAO) [5]. The property of dark energy is characterized by the equation of state $w = P/\rho$, where P is the pressure and ρ is the energy density. Dark energy has a negative pressure with w less than $-1/3$.

One of the simplest candidates of dark energy is the cosmological constant Λ with $w = -1$. The cosmological constant can arise from a vacuum energy in particle physics, but its energy scale is enormously larger than the observed energy scale of dark energy [6]. There have been many attempts to construct de Sitter vacua in supersymmetric theories. In string theory, for example, huge numbers of de Sitter vacua ($\sim 10^{500}$) can be present after the so-called flux compactification of higher-dimensional manifolds [7]. We may live in a vacuum with a tiny vacuum energy, but it is generally difficult to justify the reason for living such a specific vacuum unless some anthropic principle is introduced. So far, it is fair to say that there is no satisfactory scenario where the small energy scale of dark energy can be naturally explained by the vacuum energy related to particle physics.

If the cosmological constant problem is solved in a way that it vanishes completely, we need to find out an alternative mechanism to explain the origin of dark energy [8, 9]. Broadly speaking, we can classify dark energy models into two classes. The first one is based on a specific form of matter— such as quintessence [10, 11, 12, 13, 14, 15, 16], k-essence [17, 18], and the Chaplygin gas [19]. The second one is based on the modification of gravity at large distances (see Refs. [20] for reviews). In both classes the dark energy equation of state dynamically changes in time, by which the models can be distinguished from the Λ CDM model.

Quintessence is described by a canonical scalar field ϕ minimally coupled to gravity. A slowly varying field along a potential $V(\phi)$ can lead to the acceleration of the Universe. This mechanism is similar to slow-roll inflation in the early Universe, but the difference is that non-relativistic matter (dark matter and baryons) cannot be ignored to discuss the dynamics of dark energy correctly. Moreover, the energy scale of the quintessence potential needs to be of the order of $\rho_{\text{DE}} \approx 10^{-47} \text{ GeV}^4$ today, which is much smaller than that of the inflaton potential.

The dynamics of quintessence in the presence of non-relativistic matter has been studied in detail for many different potentials [14, 15, 16, 21, 22, 23, 24, 25]. Depending on the evolution of w , we can broadly classify quintessence models into two classes [24]: (i) thawing models and (ii) freezing models. In the first class, the field is nearly frozen by a Hubble friction during the early cosmological epoch and it starts to evolve once the field mass drops below the Hubble expansion rate. In the second class, the evolution of

the field gradually slows down because the potential tends to be shallow at late times. For the inverse power-law potential $V(\phi) = M^{4+p}\phi^{-p}$ ($p > 0$), there is so-called a tracker solution [26] along which w is nearly constant during the matter era and w starts to decrease after that. This case belongs to a subclass of freezing models. For thawing and tracker models there exist convenient analytic formulas of w [27, 28, 29, 30] employed to test the models with the data of distance measurements of SN Ia, CMB, and BAO.

The redshift-space distortions (RSD) appearing in clustering pattern of galaxies [31, 32] can provide additional constraints on the growth rate of matter perturbations δ_m . Since the evolution of δ_m is different depending on the field equation of state [33, 34, 35], it is possible to place bounds on w from the data of RSD. In fact there exist analytic formulas of δ_m and its growth rate [36], which can be used to constrain quintessence models.

In order to realize the cosmic acceleration today, the mass m_ϕ of quintessence (defined by $m_\phi^2 = d^2V(\phi)/d\phi^2$) needs to be extremely small, i.e., $|m_\phi| \lesssim H_0 \approx 10^{-33}$ eV, where H_0 is the today's Hubble parameter. In general there is a difficulty to reconcile such a ultra light mass with the energy scales appearing in particle physics [37]. Moreover, in the absence of some symmetry, the radiative corrections may disrupt the flatness of the quintessence potentials required for the cosmic acceleration [38]. However, it is not entirely hopeless to construct viable quintessence models in the framework of particle physics [39, 40, 41, 42, 43, 44, 45, 46, 47].

In this article, we review several cosmological aspects of quintessence— including its cosmological dynamics, analytic solutions of w , observational constraints, and particle physics models. The review is organized as follows. In Sec.2 we present the field equations of motion for general quintessence potentials and then proceed to the analysis of fixed points for exponential potentials. In Sec.3 we classify quintessence potentials into two classes depending on the evolution of w and then derive analytic solutions of w . These solutions are employed to put observational bounds on quintessence at the background level. In Sec.4 we derive analytic formulas for the growth rate of matter density perturbations and discuss constraints on some of quintessence models from the recent data of RSD. In Sec.5 we review theoretical models of quintessence based on supersymmetric theories. Sec. 6 is devoted to conclusions.

2. Dynamical equations of motion and exponential potentials

Let us consider quintessence in the presence of non-relativistic matter described by a barotropic perfect fluid. The total action is given by

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_{\text{pl}}^2 R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] + S_m, \quad (1)$$

where g is the determinant of the metric $g_{\mu\nu}$, M_{pl} is the reduced Planck mass, R is the Ricci scalar, S_m is the matter action. We assume that non-relativistic matter does not have a direct coupling to the quintessence field ϕ .

We study the dynamics of quintessence on the flat Friedmann-Lemaître-Robertson-Walker (FLRW) background with the line element $ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2$, where $a(t)$ is the scale factor with cosmic time t . The pressure and the energy density of quintessence are given, respectively, by $P_\phi = \dot{\phi}^2/2 - V(\phi)$ and $\rho_\phi = \dot{\phi}^2/2 + V(\phi)$, where a dot represents a derivative with respect to t . The dark energy equation of state is

$$w \equiv \frac{P_\phi}{\rho_\phi} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}. \quad (2)$$

The scalar field satisfies the continuity equation $\dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = 0$, i.e.,

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \quad (3)$$

where $H \equiv \dot{a}/a$ and $V_{,\phi} \equiv dV/d\phi$. For a matter fluid with the energy density ρ_m and the equation of state w_m , the equations of motion following from the action (1) are

$$3M_{\text{pl}}^2 H^2 = \dot{\phi}^2/2 + V(\phi) + \rho_m, \quad (4)$$

$$2M_{\text{pl}}^2 \dot{H} = - \left[\dot{\phi}^2 + (1 + w_m)\rho_m \right]. \quad (5)$$

In order to deal with the cosmological dynamics of this system, it is convenient to introduce the following dimensionless variables [14]

$$x \equiv \frac{\dot{\phi}}{\sqrt{6}M_{\text{pl}}H}, \quad y \equiv \frac{\sqrt{V(\phi)}}{\sqrt{3}M_{\text{pl}}H}. \quad (6)$$

The field density parameter $\Omega_\phi \equiv \rho_\phi/(3M_{\text{pl}}^2 H^2)$ can be expressed as $\Omega_\phi = x^2 + y^2$. From Eq. (4) the matter density parameter $\Omega_m \equiv \rho_m/(3M_{\text{pl}}^2 H^2)$ satisfies $\Omega_m = 1 - \Omega_\phi$. The field equation of state (2) reads $w = (x^2 - y^2)/(x^2 + y^2)$. We also define the effective equation of state $w_{\text{eff}} \equiv -1 - 2\dot{H}/(3H^2)$, where \dot{H}/H^2 can be evaluated from Eq. (5) as

$$\frac{\dot{H}}{H^2} = -3x^2 - \frac{3}{2}(1 + w_m)(1 - x^2 - y^2). \quad (7)$$

Taking the derivatives of x and y with respect to $N \equiv \ln a$ and using Eqs. (3) and (7), it follows that

$$\frac{dx}{dN} = -3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x \left[(1 - w_m)x^2 + (1 + w_m)(1 - y^2) \right], \quad (8)$$

$$\frac{dy}{dN} = -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y \left[(1 - w_m)x^2 + (1 + w_m)(1 - y^2) \right], \quad (9)$$

where λ is defined by $\lambda \equiv -M_{\text{pl}}V_{,\phi}/V$.

The models with constant λ corresponds to the exponential potential [13, 14, 48]

$$V(\phi) = V_0 e^{-\lambda\phi/M_{\text{pl}}}, \quad (10)$$

in which case Eqs. (8) and (9) are closed. The fixed points of this system can be derived by setting $dx/dN = 0$ and $dy/dN = 0$ [14]:

- (a) $(x, y) = (0, 0)$, $\Omega_\phi = 0$, $w_{\text{eff}} = w_m$, w is undetermined.
- (b) $(x, y) = (\pm 1, 0)$, $\Omega_\phi = 1$, $w_{\text{eff}} = w = 1$.
- (c) $(x, y) = (\lambda/\sqrt{6}, [1 - \lambda^2/6]^{1/2})$, $\Omega_\phi = 1$, $w_{\text{eff}} = w = -1 + \lambda^2/3$.

- (d) $(x, y) = (\sqrt{3/2}(1 + w_m)/\lambda, [3(1 - w_m^2)/2\lambda^2]^{1/2})$,
 $\Omega_\phi = 3(1 + w_m)/\lambda^2$, $w_{\text{eff}} = w = w_m$.

If we consider non-relativistic matter ($w_m = 0$), the matter-dominated epoch ($w_{\text{eff}} \simeq 0, \Omega_\phi \ll 1$) can be realized either by (a) or (d). The point (d) is the so-called scaling solution [13, 14], along which the ratio $\Omega_m/\Omega_\phi (\neq 0)$ remains constant. In order to realize the matter-dominated epoch by the scaling solution, we require the condition $\lambda^2 \gg 1$. On the other hand, under the condition $\lambda^2 < 2$, the epoch of cosmic acceleration ($w_{\text{eff}} < -1/3$) can be realized by the point (c). This shows that the transition from (d) to (c) is not possible, but for $\lambda^2 < 2$ the system can evolve from (a) to (c). The radiation-dominated epoch corresponds to the fixed point (a) with $w_{\text{eff}} = w_m = 1/3$.

In order to study the stabilities of the fixed points $(x, y) = (x_c, y_c)$, we consider linear perturbations δx and δy about them. Then the perturbations satisfy the following differential equations

$$\frac{d}{dN} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \mathcal{M} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{pmatrix}_{x=x_c, y=y_c}, \quad (11)$$

where $f_1(x, y)$ and $f_2(x, y)$ are the r.h.s. of Eqs. (8) and (9) respectively. If both the eigenvalues μ_1 and μ_2 of the matrix \mathcal{M} are negative, the corresponding fixed point is stable. If either μ_1 or μ_2 is negative, the point corresponds to a saddle. If both μ_1 and μ_2 are positive, the fixed point is unstable. For complex values of μ_1 and μ_2 with negative real parts, the fixed point is called a stable spiral.

The eigenvalues of the point (c) are $\mu_1 = (\lambda^2 - 6)/2$ and $\mu_2 = \lambda^2 - 3(1 + w_m)$ [14], so that it is stable under the condition $\lambda^2 < 3(1 + w_m)$ for $0 \leq w_m \leq 1$. The condition for cosmic acceleration corresponds to $\lambda^2 < 2$, in which case the point (c) is stable. The eigenvalues of the point (a) are $\mu_1 = -(3/2)(1 - w_m)$, $\mu_2 = (3/2)(1 + w_m)$, so that it is a saddle for $0 \leq w_m \leq 1$. This means that, for $\lambda^2 < 2$, the solution eventually exits the point (a) to approach the attractor point (c).

The dark energy equation of state w for the point (a) is undetermined, but in the realistic Universe, x and y are not exactly 0. The early evolution of w depends on the initial conditions of x and y . If $x^2 \gg y^2$ and $x^2 \ll y^2$, we have $w \simeq 1$ and $w \simeq -1$ respectively. Finally the solution approaches the constant value $w = -1 + \lambda^2/3$. Since w dynamically changes in this way, the quintessence model with the exponential potential is observationally distinguishable from the Λ CDM model.

For quintessence models in which λ is not constant, Eqs. (8) and (9) are not closed. In such cases the situation is more involved, but it is possible to derive analytic solutions of w by classifying quintessence potentials according to the evolution of w . In the next section we shall address this issue.

3. Classification of quintessence models and observational constraints

In order to study the evolution of w for the models with varying λ , we derive the differential equations for w and Ω_ϕ . Using Eqs. (7)-(9), we obtain

$$w' = (w - 1)[3(1 + w) - \lambda\sqrt{3(1 + w)\Omega_\phi}], \quad (12)$$

$$\Omega'_\phi = -3(w - w_m)\Omega_\phi(1 - \Omega_\phi), \quad (13)$$

where a prime represents a derivative with respect to $N = \ln a$. Introducing the quantity $\Gamma \equiv VV_{,\phi\phi}/V_{,\phi}^2$, the parameter λ obeys

$$\lambda' = -\sqrt{3(1 + w)\Omega_\phi}(\Gamma - 1)\lambda^2. \quad (14)$$

The evolution of w is different depending on the quintessence potentials and the initial conditions. In what follows we discuss three qualitatively different cases: (i) tracking freezing models, (iii) scaling freezing models, and (iii) thawing models. In freezing models the potential tends to be shallow at late times, which results in the decrease of w . In thawing models the mass of the field becomes smaller than H only recently, so that the deviation of w from -1 occurs at late times.

3.1. Tracking freezing models

For the field density parameter satisfying the relation

$$\Omega_\phi = 3(1 + w)/\lambda^2, \quad (15)$$

w is constant from Eq. (12). If $w = w_m$, then Ω_ϕ is constant from Eq. (13) and hence λ is constant. This corresponds to the scaling solution (d) discussed in Sec. 2. Since the scaling solution is stable for $\lambda^2 > 3(1 + w_m)$ [14], it does not exit to the fixed point (c). If λ decreases in time, the system can enter the epoch of cosmic acceleration. From Eq. (14) this condition translates into

$$\Gamma > 1. \quad (16)$$

The solution (15) satisfying the condition (16) is called a tracker [26], along which Ω_ϕ increases and hence $w < w_m$. From Eq. (15) we have the relation $\Omega'_\phi/\Omega_\phi = -2\lambda'/\lambda$. Using Eqs. (13) and (14) under the condition $\Omega_\phi \ll 1$, the constant equation of state along the tracker is [26]

$$w = w_{(0)} \equiv \frac{w_m - 2(\Gamma - 1)}{2\Gamma - 1}. \quad (17)$$

For example, let us consider the inverse power-law potential [13]

$$V(\phi) = M^{4+p}\phi^{-p}, \quad (18)$$

where M and p (> 0) are constants. Since in this case $\Gamma = 1 + 1/p > 1$, the tracking condition (16) is satisfied. The constant equation of state (17) is $w_{(0)} = (p w_m - 2)/(p + 2)$ and hence $w_{(0)} = -2/(p + 2)$ during the matter era. With the growth of Ω_ϕ , w starts to decrease from $w_{(0)}$. Hence the tracker belongs to the class of freezing models.

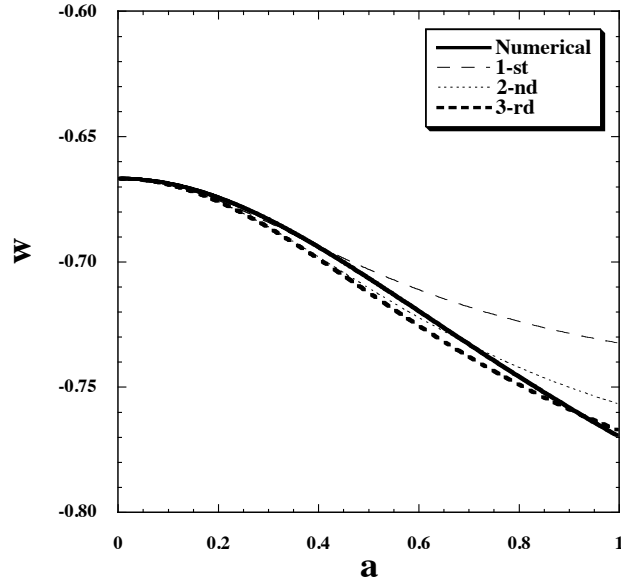


Figure 1. The field equation of state w versus a for the tracker solution. This case corresponds to the inverse power-law potential $V(\phi) = M^5\phi^{-1}$. The solid curve is derived by solving Eqs. (12) and (13) numerically, whereas other curves show the 1-st, 2-nd, 3-rd order analytic solutions (21).

The analytic solution (17) is valid in the regime $\Omega_\phi \ll 1$. We take into account the variation of w by dealing with Ω_ϕ as a perturbation to the 0-th order solution (17). From Eqs. (12) and (14), the perturbation δw around $w_{(0)}$ obeys [30]

$$a^2 \frac{d^2 \delta w}{da^2} + \frac{5 - 6w_{(0)}}{2} a \frac{d\delta w}{da} + \frac{9}{2} (1 - w_{(0)}) \delta w - \frac{9}{2} w_{(0)} (1 - w_{(0)}^2) \Omega_\phi(a) = 0, \quad (19)$$

where we assumed that Γ is nearly constant. For $\Omega_\phi(a)$ we use the 0-th order solution

$$\Omega_\phi(a) = \frac{\Omega_{\phi 0} a^{-3w_{(0)}}}{\Omega_{\phi 0} a^{-3w_{(0)}} + 1 - \Omega_{\phi 0}}, \quad (20)$$

where $\Omega_{\phi 0}$ is the today's value of Ω_ϕ . Substituting Eq. (20) into Eq. (19), we obtain the following integrated solution [30]

$$w(a) = w_{(0)} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} w_{(0)} (1 - w_{(0)}^2)}{1 - (n+1)w_{(0)} + 2n(n+1)w_{(0)}^2} \left(\frac{\Omega_\phi(a)}{1 - \Omega_\phi(a)} \right)^n. \quad (21)$$

In Fig. 1 we plot the evolution of w derived by the analytic solution (21) for the inverse power-law potential $V(\phi) = M^5\phi^{-1}$. Each curve corresponds to the 1-st, 2-nd, 3-rd order solution, whereas the solid curve is derived by solving Eqs. (12)-(14) numerically. We find that the analytic solution up to 3-rd order shows good agreement with the full numerical result. The analytic expression of w is parametrized by two parameters $w_{(0)}$ and $\Omega_{\phi 0}$ alone.

The observational constraints on the tracker model have been carried out in Refs. [30, 49, 50]. In addition to the SN Ia data, the distance measurements of the

CMB and BAO peaks provide the information of the background expansion history from the recombination epoch to today. From the joint data analysis of Union 2.1 [51], WMAP7 [52], and BAO (SDSS7 [53] and BOSS [54]), the tracker equation of state during the matter era is constrained to be $w_{(0)} < -0.964$ (95% CL) under the prior $w_{(0)} > -1$ [50]. For the potential (18) this bound translates into $p < 0.075$. In Ref. [50] it was found that the best-fit corresponds to $w_{(0)} = -1$, i.e., the Λ CDM. If we do not put the prior $w_{(0)} > -1$, the best-fit model parameters are found to be $w_{(0)} = -1.097$ and $\Omega_{\phi 0} = 0.717$. With the BOSS BAO data [54] the phantom equation of state ($w_{(0)} < -1$) is particularly favored, but this is not the regime of quintessence.

3.2. Scaling freezing models

The scaling solution [13, 14] can be regarded as a special case of a tracker along which $\Omega_\phi = 3(1+w)/\lambda^2$ is constant. During the matter era, $w = w_m = 0$ and hence $\Omega_\phi = 3/\lambda^2$. Since λ is constant, $\Gamma = 1$ from Eq. (14). This case corresponds to the exponential potential (10), but the system does not enter the phase of cosmic acceleration because the field equation of state is the same as that of the background fluid.

This problem can be alleviated by considering the double exponential potential [55]

$$V(\phi) = V_1 e^{-\lambda_1 \phi/M_{\text{Pl}}} + V_2 e^{-\lambda_2 \phi/M_{\text{Pl}}}, \quad (22)$$

where λ_i and V_i ($i = 1, 2$) are constants (see Refs. [56, 57, 58] for related potentials). For the parameters satisfying the conditions $\lambda_1 \gg 1$ and $\lambda_2 \lesssim 1$, the solution first enters the scaling regime characterized by $\Omega_\phi = 3(1+w_m)/\lambda_1^2$. During the radiation era ($w_m = 1/3$) the constraint coming from the big bang nucleosynthesis gives the bound $\Omega_\phi < 0.045$ (95 % CL) [59], which translates into the condition $\lambda_1 > 9.4$. The scaling matter era ($\Omega_\phi = 3/\lambda_1^2$, $w = 0$) is followed by the epoch of cosmic acceleration driven by another exponential potential $V_2 e^{-\lambda_2 \phi/M_{\text{Pl}}}$. In this case the solution finally approaches the fixed point (c) discussed in Sec. 2.

The onset of the transition from the scaling matter era to the epoch of cosmic acceleration depends on the parameters λ_1 , λ_2 , and V_2/V_1 . The transition redshift is not very sensitive to the choice of V_2/V_1 , so we can set $V_2 = V_1$ without loss of generality. In Fig. 2 we show the numerical evolution of w for $\lambda_2 = 0$ with three different values of λ_1 . For larger λ_1 the transition to $w = -1$ occurs earlier.

The above variation of w can be accommodated by using the parametrization [60]

$$w(a) = w_f + \frac{w_p - w_f}{1 + (a/a_t)^{1/\tau}}, \quad (23)$$

where w_p and w_f are asymptotic values of w in the past and future respectively, a_t is the scale factor at the transition, and τ describes the transition width (see Refs. [61] for early related works). The scaling solution during the matter-dominated epoch corresponds to $w_p = 0$. For $\lambda_2 = 0$ we have $w_f = -1$, in which case Eq. (23) reduces to $w(a) = -1 + [1 + (a/a_t)^{1/\tau}]^{-1}$. As we see in Fig. 2, the parametrization (23) fits the numerical solutions of w very well for appropriate choices of a_t and τ . For the models with $\lambda_2 = 0$ the transition width is around $\tau \approx 0.33$, while a_t depends on λ_1 .

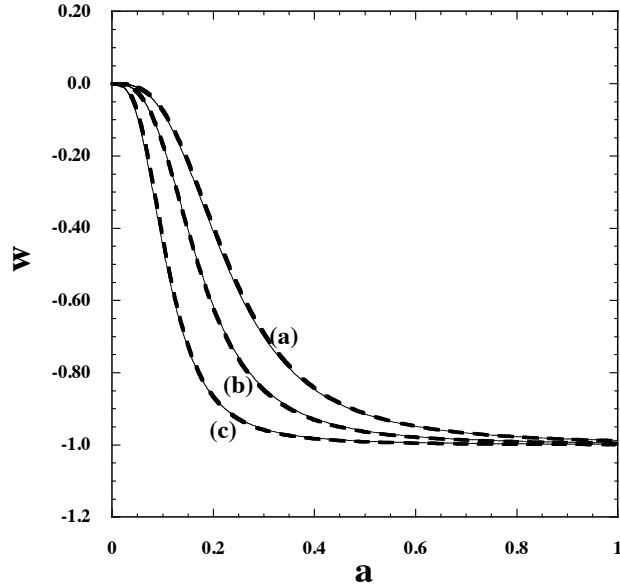


Figure 2. The field equation of state w versus a for the potential (22) with (a) $\lambda_1 = 10$, $\lambda_2 = 0$, (b) $\lambda_1 = 15$, $\lambda_2 = 0$, and (c) $\lambda_1 = 30$, $\lambda_2 = 0$. The solid curves are the numerically integrated solutions, whereas the dashed curves show the results derived from the parametrization (23) with $w_p = 0$ and $w_f = -1$. Each dashed curve corresponds to (a) $a_t = 0.23$, $\tau = 0.33$, (b) $a_t = 0.17$, $\tau = 0.33$, and (c) $a_t = 0.11$, $\tau = 0.32$.

In Ref. [50] the joint data analysis of Union 2.1, WMAP7, and BAO (SDSS7 and BOSS) was carried out by fixing $\tau = 0.33$. The transition redshift was found to be $a_t < 0.23$ (95 % CL). The case (a) shown in Fig. 2 is the marginal one where the model is within the 2σ observational contour. This shows that w needs to approach -1 in the early cosmological epoch. For $\lambda_2 > 0$ the likelihood analysis was also performed in Ref. [50] by numerically solving the field equations of motion with suitable initial conditions. The model parameters are constrained to be $\lambda_1 > 11.7$, $\lambda_2 < 0.539$, and $0.256 < \Omega_{m0} < 0.279$ (95 % CL). The models with $\lambda_2 \gtrsim 0.5$ are disfavored because the deviation of w from -1 tends to be significant.

In k-essence models where the Lagrangian P depends on the field ϕ and the kinetic energy $X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$ [17, 18], the condition for the existence of scaling solutions restricts the Lagrangian to the form $P(\phi, X) = Xg(Xe^{\lambda\phi/M_{\text{Pl}}})$ [62, 63], where g is an arbitrary function in terms of $Y \equiv Xe^{\lambda\phi/M_{\text{Pl}}}$. The quintessence with the exponential potential ($P = X - ce^{-\lambda\phi/M_{\text{Pl}}}$) corresponds to the choice $g(Y) = 1 - c/Y$, whereas the choice $g(Y) = -1 + cY$ gives rise to the dilatonic ghost condensate model $P = -X + ce^{\lambda\phi/M_{\text{Pl}}}X^2$ [62]. For the multi-field scaling Lagrangian given by

$$P(\phi_i, X_i) = \sum_{i=1}^n X_i g(X_i e^{\lambda_i \phi_i / M_{\text{Pl}}}), \quad (24)$$

it was shown [64] that a phenomenon called *assisted inflation* [65] occurs with the

effective slope $\lambda_{\text{eff}} = (\sum_{i=1}^n 1/\lambda_i^2)^{-1/2}$, irrespective of the form of g . In the presence of multiple fields, the scaling matter era can be followed by the epoch of cosmic acceleration even if the individual field is unable to lead to the accelerated expansion. In Refs. [66, 67] the cosmological dynamics of assisted dark energy was studied in detail.

3.3. Thawing models

In thawing models the field is nearly frozen by the Hubble friction in the early cosmological epoch. In this regime one has $w \simeq -1$, which corresponds to one of the fixed points of (12). The representative model of this class is characterized by the potential of the pseudo-Nambu-Goldstone boson (PNGB) [39]:

$$V(\phi) = \mu^4 [1 + \cos(\phi/f_a)] , \quad (25)$$

where μ and f_a are constants having a dimension of mass.

Let us consider the case in which the field initially exists around $\phi = \phi_i$ and then it starts to evolve after the field mass drops below H . We expand the potential $V(\phi)$ around $\phi = \phi_i$ up to second order, as $V(\phi) = \sum_{n=0}^2 V^{(n)}(\phi_i) (\phi - \phi_i)^n/n!$. Using the approximation $P_\phi \simeq -\rho_\phi \simeq -V(\phi_i)$ and redefining the field $u = (\phi - \phi_i)a^{3/2}$, Eq. (3) reads [28, 29]

$$\ddot{u} - \omega^2 u \simeq -a^{3/2} V_{,\phi}(\phi_i) , \quad \omega = \left[\frac{3}{4} \frac{V(\phi_i)}{M_{\text{pl}}^2} - V_{,\phi\phi}(\phi_i) \right]^{1/2} , \quad (26)$$

where we assumed $3V(\phi_i)/(4M_{\text{pl}}^2) > V_{,\phi\phi}(\phi_i)$. Provided that $|w + 1| \ll 1$ the evolution of the scale factor can be approximated as that of the Λ CDM model, in which case $\dot{a}^2 = H_0^2 a^2 [\Omega_{\phi 0} + (1 - \Omega_{\phi 0})a^{-3}]$ from Eq. (4). Integration of this equation gives

$$a(t) = \left(\frac{1 - \Omega_{\phi 0}}{\Omega_{\phi 0}} \right)^{1/3} \sinh^{2/3}(t/t_\Lambda) , \quad t_\Lambda = \frac{2M_{\text{pl}}}{\sqrt{3V(\phi_i)}} . \quad (27)$$

Substituting Eq. (27) into Eq. (26), we obtain the following solution

$$u(t) = A \sinh(\omega t) + B \cosh(\omega t) + \sqrt{\frac{1 - \Omega_{\phi 0}}{\Omega_{\phi 0}}} \frac{V_{,\phi}(\phi_i) t_\Lambda^2}{\omega^2 t_\Lambda^2 - 1} \sinh(t/t_\Lambda) , \quad (28)$$

which is valid for $\omega t_\Lambda \neq 1$ (i.e., $V_{,\phi\phi}(\phi_i) \neq 0$). The integration constants A and B are determined by the initial conditions $\phi(0) = \phi_i$ and $\dot{\phi}(0) = 0$. Then, the solution is

$$\phi(t) = \phi_i + \frac{V_{,\phi}(\phi_i)}{V_{,\phi\phi}(\phi_i)} \left[\frac{\sinh(\omega t)}{\omega t_\Lambda \sinh(t/t_\Lambda)} - 1 \right] . \quad (29)$$

Since $w + 1 \simeq \dot{\phi}^2(t)/V(\phi_i)$ under the approximation $\rho_\phi \simeq V(\phi_i)$, it follows that

$$w + 1 \simeq \frac{3}{4} \left[\frac{V_{,\phi}(\phi_i)}{\omega t_\Lambda V_{,\phi\phi}(\phi_i)} \right]^2 \left[\frac{\omega t_\Lambda \cosh(\omega t) \sinh(t/t_\Lambda) - \sinh(\omega t) \cosh(t/t_\Lambda)}{\sinh^2(t/t_\Lambda)} \right]^2 . \quad (30)$$

The field equation of state can be written as a function of a by using the value w_0 today ($t = t_0$, $a = 1$). Introducing the dimensionless variables

$$K \equiv \omega t_\Lambda = \sqrt{1 - \frac{4}{3} \frac{M_{\text{pl}}^2 V_{,\phi\phi}(\phi_i)}{V(\phi_i)}} , \quad F(a) \equiv \sqrt{1 + [(\Omega_{\phi 0})^{-1} - 1]a^{-3}} , \quad (31)$$

we obtain the relation $\omega t = K \sinh^{-1} \sqrt{a^3 \Omega_{\phi 0} / (1 - \Omega_{\phi 0})}$ from Eq. (27). Then Eq. (30) reads [28, 29]

$$w(a) = -1 + (1 + w_0) a^{3(K-1)} \mathcal{F}(a), \quad (32)$$

where

$$\mathcal{F}(a) = \left[\frac{(K - F(a))(F(a) + 1)^K + (K + F(a))(F(a) - 1)^K}{(K - \Omega_{\phi 0}^{-1/2})(\Omega_{\phi 0}^{-1/2} + 1)^K + (K + \Omega_{\phi 0}^{-1/2})(\Omega_{\phi 0}^{-1/2} - 1)^K} \right]^2. \quad (33)$$

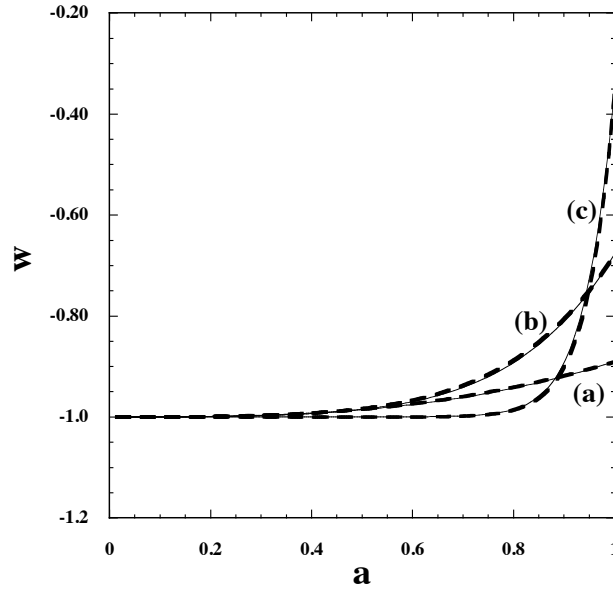


Figure 3. The field equation of state w versus a for the potential (25) with (a) $f_a/M_{\text{pl}} = 0.5$, $\phi_i/f_a = 0.5$ ($K = 1.9$), (b) $f_a/M_{\text{pl}} = 0.3$, $\phi_i/f_a = 0.25$ ($K = 2.9$), and (c) $f_a/M_{\text{pl}} = 0.1$, $\phi_i/f_a = 7.6 \times 10^{-4}$ ($K = 8.2$). The solid curves correspond to numerically integrated solutions, whereas the bald dashed curves show the results derived from the analytic solution (32) with $\Omega_{\phi 0} = 0.73$.

The field equation of state (32) is expressed in terms of the three parameters w_0 , $\Omega_{\phi 0}$, and K . The quantity K is related to the field mass squared $m_\phi^2 = V_{,\phi\phi}$. For the potential (25) we have $K > 1$ for $0 < \phi_i/f_a < \pi/2$ and $K < 1$ for $\pi/2 < \phi_i/f_a < \pi$, respectively. If $4M_{\text{pl}}^2 V_{,\phi\phi}(\phi_i)/(3V(\phi_i)) > 1$ (i.e., $K^2 < 0$), we can derive the similar expression of w by setting $K = i\hat{K}$ and $\hat{K} = \sqrt{4M_{\text{pl}}^2 V_{,\phi\phi}(\phi_i)/(3V(\phi_i)) - 1}$ [29]. For a phantom field and a thawing k-essence field, analytic solutions of w similar to (32) were derived in Refs. [68, 69].

In Fig. 3 we plot numerically integrated solutions of w as well as analytic solutions based on (32) for three different values of K . As long as $K \lesssim 10$ and $w_0 \lesssim -0.3$, the analytic estimation of (32) is sufficiently trustable. For larger K , the field mass squared $|m_\phi^2|$ increases and hence the variation of w around today is more significant. For the validity of the Taylor expansion used to derived the analytic solution (32), we require

the condition $|K - 1| < \mathcal{O}(1)$. For the models with $V_{,\phi\phi} > 0$, the analytic solutions start to lose the accuracy for K smaller than 0.5.

The observational constraints on thawing models have been carried out in Refs. [28, 69, 70, 71, 50]. If the three parameters w_0 , $\Omega_{\phi 0}$, and K are varied in the likelihood analysis with the prior $0.1 < K < 10$, the constraints on K are generally weak. After the marginalization over K without any prior on w_0 , Chiba *et al.* [50] derived the bounds $-2.18 < w_0 < -0.893$ and $0.703 < \Omega_{\phi 0} < 0.735$ (95 % CL). If we put the prior $w_0 > -1$, the field equation of state is constrained to be $w_0 < -0.849$ (68 % CL) and $w_0 < -0.695$ (95 % CL). Although there is no statistical evidence that the models with $w_0 > -1$ are favored over the Λ CDM, the thawing models with $-1 < w_0 < -0.7$ are not yet ruled out observationally.

4. Constraints from the large-scale structure

In addition to the background observational constraints discussed in Sec. 3, the quintessence models can be distinguished from the Λ CDM by considering the evolution of cosmological perturbations. The peculiar velocity of inward collapse motion of the large-scale structure is directly related to the growth rate of the matter density contrast δ_m . Then the measurement from redshift-space distortions (RSD) of clustering pattern of galaxies can constrain the growth history the large-scale structure. The galaxy redshift surveys provide bounds on the growth rate $f(z)$ or $f(z)\sigma_8(z)$ in terms of the redshift $z = 1/a - 1$, where $f = d \ln \delta_m / d \ln a$ and σ_8 is the rms amplitude of δ_m at the comoving scale $8 h^{-1}$ Mpc (h is the normalized Hubble constant $H_0 = 100 h \text{ km sec}^{-1} \text{ Mpc}^{-1}$). It is then convenient to derive analytic solutions of $f(z)$ and $f(z)\sigma_8(z)$ to confront quintessence models with the observations of RSD.

Let us consider scalar metric perturbations Φ about the flat FLRW background. Neglecting the anisotropic stress, the metric in the Newtonian gauge is given by [72]

$$ds^2 = -(1 - 2\Phi) dt^2 + a^2(t)(1 + 2\Phi)d\mathbf{x}^2. \quad (34)$$

We decompose the matter density ρ_m into the background and inhomogeneous parts, as $\rho_m = \bar{\rho}_m(t) + \delta\rho_m(t, \mathbf{x})$. We also define

$$\delta_m \equiv \delta\rho_m / \bar{\rho}_m, \quad \theta_m \equiv \nabla^2 v_m / (aH), \quad (35)$$

where v_m is the rotational-free velocity potential of non-relativistic matter. In the Fourier space the matter perturbations satisfy [72, 73]

$$\delta'_m = -3\Phi' - \theta_m, \quad (36)$$

$$\theta'_m + \left(2 + \frac{H'}{H}\right) \theta_m = -\left(\frac{k}{aH}\right)^2 \Phi, \quad (37)$$

where k is a comoving wave number and a prime represents a derivative with respect to $N = \ln a$. Taking the N -derivative of Eq. (36) and using Eq. (37), we obtain

$$\delta''_m + \left(2 + \frac{H'}{H}\right) \delta'_m - \left(\frac{k}{aH}\right)^2 \Phi = -3 \left[\Phi'' + \left(2 + \frac{H'}{H}\right) \Phi' \right]. \quad (38)$$

Provided that quintessence does not cluster, we can neglect the contribution of quintessence perturbations relative to matter perturbations. In this case the Poisson equation is approximately given by

$$\left(\frac{k}{aH}\right)^2 \Phi \simeq \frac{3}{2} \Omega_m \hat{\delta}_m, \quad (39)$$

where $\hat{\delta}_m \equiv \delta_m + 3(aH/k)^2 \theta_m$ is the rest-frame gauge-invariant density perturbation. For the modes deep inside the Hubble radius ($k \gg aH$) relevant to the large-scale structure, the r.h.s. of Eq. (38) can be neglected relative to the l.h.s. of it. Using the approximate relation $\hat{\delta}_m \simeq \delta_m$ together with Eqs. (7) and (39), we find that Eq. (38) reduces to

$$\delta_m'' + \frac{1}{2} (1 - 3w\Omega_\phi) \delta_m' - \frac{3}{2} \Omega_m \delta_m \simeq 0. \quad (40)$$

In the following we derive analytic solutions for the growth rate f and $f\sigma_8$ as functions of the redshift z . Since $\Omega_\phi \ll 1$ at the early cosmological epoch, we expand the quintessence equation of state in terms of Ω_ϕ , as

$$w = w_0 + \sum_{n=1}^{\infty} w_n (\Omega_\phi)^n. \quad (41)$$

We introduce the growth index γ , as $f = \delta_m'/\delta_m = (\Omega_m)^\gamma = (1 - \Omega_\phi)^\gamma$ [74, 33, 34]. Using Eq. (13) with $w_m = 0$, Eq. (40) reads [33]

$$3w\Omega_\phi(1 - \Omega_\phi) \ln(1 - \Omega_\phi) \frac{d\gamma}{d\Omega_\phi} = \frac{1}{2} - \frac{3}{2} w(1 - 2\gamma)\Omega_\phi + (1 - \Omega_\phi)^\gamma - \frac{3}{2} (1 - \Omega_\phi)^{1-\gamma}. \quad (42)$$

The solution of Eq. (42) can be derived by expanding γ in terms of Ω_ϕ , as $\gamma = \gamma_0 + \sum_{n=1}^{\infty} \gamma_n (\Omega_\phi)^n$. On using the expansion (41) as well, we obtain [35, 36]

$$\gamma = \frac{3(1 - w_0)}{5 - 6w_0} + \frac{3(1 - w_0)(2 - 3w_0) + 2w_1(5 - 6w_0)}{(5 - 6w_0)^2(5 - 12w_0)} \Omega_\phi + \mathcal{O}(\Omega_\phi^2). \quad (43)$$

If $w_0 = -1$ and $w_1 = 0$, then we have $\gamma \simeq 0.545 + 7.29 \times 10^{-3} \Omega_\phi$. Since the second term is much smaller than the first one, γ is nearly constant. Even for the models with $w_0 = -1$ and $w_1 = 0.3$ (where the value of w today is around -0.8), the variation of γ is small: $\gamma \simeq 0.545 + 1.21 \times 10^{-2} \Omega_\phi$.

The relation $\delta_m'/\delta_m = (1 - \Omega_\phi)^\gamma$ can be written in the form

$$\frac{d}{d\Omega_\phi} \ln \delta_m = -\frac{(1 - \Omega_\phi)^{\gamma-1}}{3w\Omega_\phi}. \quad (44)$$

Under the approximation that γ is constant, the term $(1 - \Omega_\phi)^{\gamma-1}$ can be expanded around $\Omega_\phi = 0$, as

$$(1 - \Omega_\phi)^{\gamma-1} = 1 + \sum_{n=1}^{\infty} \alpha_n (\Omega_\phi)^n, \quad \alpha_n = \frac{(-1)^n}{n!} \prod_{i=1}^n (\gamma - i). \quad (45)$$

We also expand $1/w$ in the form $1/w = (1/w_0)[1 + \sum_{n=1}^{\infty} \beta_n (\Omega_\phi)^n]$, where β_n can be expressed in terms of w_i ($i = 0, 1, 2, \dots$). Then Eq. (44) is written as

$$\frac{d}{d\Omega_\phi} \ln \delta_m = -\frac{1}{3w_0\Omega_\phi} \left[1 + \sum_{n=1}^{\infty} c_n (\Omega_\phi)^n \right], \quad c_n = \sum_{i=0}^n \alpha_{n-i} \beta_i, \quad (46)$$

with $\alpha_0 = \beta_0 = 1$. Integration of Eq. (46) leads to the following solution

$$\delta_m = \delta_{m0} \exp \left\{ \frac{1}{3w_0} \left[\ln \frac{\Omega_{\phi 0}}{\Omega_{\phi}} + \sum_{n=1}^{\infty} \frac{c_n}{n} ((\Omega_{\phi 0})^n - (\Omega_{\phi})^n) \right] \right\}, \quad (47)$$

where δ_{m0} is the today's value of δ_m .

The perturbation δ_g of galaxies is related to δ_m , as $\delta_g = b\delta_m$, where b is a bias factor. The galaxy power spectrum $\mathcal{P}_g(\mathbf{k})$ in the redshift space can be expressed as [31, 32]

$$\mathcal{P}_g(\mathbf{k}) = \mathcal{P}_{gg}(\mathbf{k}) - 2\mu^2 \mathcal{P}_{g\theta}(\mathbf{k}) + \mu^4 \mathcal{P}_{\theta\theta}(\mathbf{k}), \quad (48)$$

where $\mu = \mathbf{k} \cdot \mathbf{r}/(kr)$ is the cosine of the angle of the momentum vector \mathbf{k} to the line of sight (vector \mathbf{r}), $\mathcal{P}_{gg}(\mathbf{k})$ and $\mathcal{P}_{\theta\theta}(\mathbf{k})$ are the real space power spectra of galaxies and θ , respectively, and $\mathcal{P}_{g\theta}(\mathbf{k})$ is the cross power spectrum of galaxy- θ fluctuations in real space. Neglecting the variation of Φ in Eq. (36), it follows that

$$\theta_m \simeq -f\delta_m. \quad (49)$$

The three spectra \mathcal{P}_{gg} , $\mathcal{P}_{g\theta}$, and $\mathcal{P}_{\theta\theta}$ depend on $(b\delta_m)^2$, $(b\delta_m)(f\delta_m)$, and $(f\delta_m)^2$, respectively. Provided that the growth of perturbations is scale-independent, the constraints on $b\delta_m$ and $f\delta_m$ at some scale translate into those on $b\sigma_8$ and $f\sigma_8$. The quantity $f\sigma_8$ is useful because it does not include the bias factor b .

Normalizing δ_{m0} in terms of $\sigma_8(z=0)$ in Eq. (47), we obtain [36]

$$f(z)\sigma_8(z) = (1-\Omega_{\phi})^{\gamma} \sigma_8(z=0) \exp \left\{ \frac{1}{3w_0} \left[\ln \frac{\Omega_{\phi 0}}{\Omega_{\phi}} + \sum_{n=1}^{\infty} \frac{c_n}{n} ((\Omega_{\phi 0})^n - (\Omega_{\phi})^n) \right] \right\}. \quad (50)$$

The 0-th order solution of Ω_{ϕ} corresponds to $w = w_0$, in which case $\Omega_{\phi}^{(0)} = \Omega_{\phi 0}(1+z)^{3w_0}/[1 - \Omega_{\phi 0} + \Omega_{\phi 0}(1+z)^{3w_0}]$. Using the iterative solution $w = w_0 + w_1\Omega_{\phi}^{(0)}$, we obtain the first-order solution

$$\Omega_{\phi}^{(1)} = \frac{\Omega_{\phi 0}(1+z)^{3w_0}[1 - \Omega_{\phi 0} + \Omega_{\phi 0}(1+z)^{3w_0}]^{w_1/w_0}}{1 - \Omega_{\phi 0} + \Omega_{\phi 0}(1+z)^{3w_0}[1 - \Omega_{\phi 0} + \Omega_{\phi 0}(1+z)^{3w_0}]^{w_1/w_0}}. \quad (51)$$

We can continue the similar iterative processes, but it is practically sufficient to exploit the 1-st order solution (51) for the evaluation of Ω_{ϕ} in Eq. (50).

There exists a quintessence potential in which w is constant [75, 36] (see also Ref. [8]). In this case we have 3 free parameters w_0 , $\Omega_{\phi 0}$, $\sigma_8(z=0)$ in the expression of $f(z)\sigma_8(z)$. In tracking quintessence models the coefficients w_n ($n \geq 1$) are expressed in terms of $w_0 = w_{(0)}$, so there are also 3 free parameters w_0 , $\Omega_{\phi 0}$, and $\sigma_8(z=0)$. In these cases it was shown that the analytic result (50) up to 7-th order terms of c_n is sufficiently accurate to reproduce full numerical solutions in high precision [36]. In thawing quintessence models, when the variation of w is fast at late times, the analytic solution (50) is not very accurate unless higher-order terms of c_n are taken into account.

In constant w models the observational data of RSD up to 2012 place the bound $-1.245 < w < -0.347$ (68 % CL), whereas in the tracking models the tracker equation of state is constrained to be $-1.288 < w_{(0)} < -0.214$ (68 % CL). Although these constraints are still weak, this situation will be improved in future high-precision measurements.

5. Particle physics models of quintessence

There have been many attempts to construct particle physics model of quintessence in the framework of supersymmetric theories. Binetruy [76] showed that the inverse power-law potential (18) appears in a globally supersymmetric $SU(N_c)$ gauge theory with N_c colors and the condensation of N_f flavors. In this theory the power p in Eq. (18) is given by $p = 2(N_c + N_f)/(N_c - N_f)$, which is larger than 2 under the condition $N_c \geq N_f > 0$. Since p is constrained to be smaller than 0.075 [50], this scenario is not compatible with the current observational data.

In the presence of gravity, any globally supersymmetric theory reduces to a locally supersymmetric supergravity theory. In supergravity the four-dimensional effective action is given by [77]

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{\text{pl}}^2}{2} R - K_{ij^*} \partial_\mu \varphi^i \partial^\mu \varphi^{j^*} - V(\varphi, \varphi^*) \right], \quad (52)$$

where φ are chiral scalar fields, and K^{ij^*} is an inverse of the derivative of the so-called Kähler potential K , i.e., $K_{ij^*} \equiv \partial^2 K / \partial \varphi^i \partial \varphi^{j^*}$. The effective cosmological constant V is expressed in terms of K and the superpotential W , as

$$V(\varphi, \varphi^*) = e^{K/M_{\text{pl}}^2} [D_i W (K^{ij^*}) (D_j W)^* - 3|W|^2/M_{\text{pl}}^2], \quad (53)$$

where $D_i W \equiv \partial W / \partial \varphi^i + (W/M_{\text{pl}}^2) \partial K / \partial \varphi^i$.

The last term in Eq. (53) is negative and hence this can be an obstacle to realize a positive vacuum energy required for dark energy. For example, Brax and Martin [78] chose a superpotential $W = \Lambda^{3+\alpha} \varphi^{-\alpha}$ (motivated by the fermion condensate gauge theory mentioned above) and a flat Kähler potential $K = \varphi \varphi^*$, but in this case the potential V becomes negative for $\phi \sim M_{\text{pl}}$. This problem can be avoided by imposing $\langle W \rangle = 0$ [78], but such a constraint is generally difficult to be compatible with the models of supersymmetry breaking. The Kähler potential of the form $K = -M_{\text{pl}}^2 \ln[(\varphi + \varphi^*)/M_{\text{pl}}]$, which is present at tree level for both the dilaton and moduli fields in string theory, can allow the possibility of canceling the negative term $(-3|W|^2/M_{\text{pl}}^2)$. Introducing a new field $\phi = (M_{\text{pl}}/\sqrt{2}) \ln(\varphi/M_{\text{pl}})$ in this case, the kinetic term in the action (52) reduces to the canonical form $\mathcal{L}_{\text{kin}} = -\partial^\mu \phi \partial_\mu \phi / 2$. For the choice $W = \Lambda^{3+\alpha} \varphi^{-\alpha}$, the potential (53) reads [79]

$$V(\phi) = M^4 e^{-\sqrt{2}\beta \phi / M_{\text{pl}}}, \quad (54)$$

where $\beta \equiv 2\alpha + 1$ and $M^4 \equiv M_{\text{pl}}^{-\beta-1} \Lambda^{\beta+5} (\beta^2 - 3)/2$. The positivity of the potential requires the condition $\beta > \sqrt{3}$. Then the slope of the exponential potential, $\lambda \equiv \sqrt{2}\beta$, satisfies the condition $\lambda > \sqrt{6}$. In this case there exists a scaling solution along which $\Omega_\phi = 3(1 + w_m)/\lambda^2$ is constant with $w = w_m$, but the potential needs to be modified at late times to realize the cosmic acceleration.

Copeland *et al.* [79] tried to construct a viable quintessence potential by choosing

$$K = M_{\text{pl}}^2 [\ln(\varphi + \varphi^*)/M_{\text{pl}}]^2, \quad W = \Lambda^{3+\alpha} \varphi^{-\alpha}, \quad (55)$$

where the field φ is assumed to be real. The kinetic term becomes canonical by introducing a new scalar field, $\phi \equiv \int \sqrt{2K_{\varphi\varphi^*}} d\varphi = -(2M_{\text{pl}}/3) [1 - \ln(2\varphi/M_{\text{pl}})]^{3/2}$. The field potential is given by

$$V = m^4 [2Y^2 + (4\alpha - 7)Y + 2(\alpha - 1)^2] e^{(1-Y)^2 - 2\alpha(1-Y)}/Y, \quad (56)$$

where $m^4 \equiv 2^{2\alpha} M_{\text{pl}}^{-2-2\alpha} \Lambda^{6+2\alpha}$ and

$$Y \equiv 1 - \ln(2\varphi/M_{\text{pl}}) = [-(3/2)(\phi/M_{\text{pl}})]^{2/3}. \quad (57)$$

The field exists in the region $-\infty < \phi < 0$, which corresponds to $0 < Y < \infty$. For $|\phi| \ll M_{\text{pl}}$ and $|\phi| \gg M_{\text{pl}}$, the potential behaves as $V \propto (-\phi)^{-2/3}$ and $V \propto (-\phi)^{2/3} e^{(-\phi/M_{\text{pl}})^{4/3}}$ respectively. In the intermediate region there exists a potential minimum with a positive energy density. For the initial conditions satisfying $|\phi| \ll M_{\text{pl}}$ the quintessence potential is approximately given by $V(\phi) \propto (-\phi)^{-2/3}$ in the early cosmological epoch, so that the field exhibits a tracking behavior. If the field is initially in the region $|\phi| \gg M_{\text{pl}}$, the contribution of the exponential potential is important. In this case a scaling-like behavior can be realized during the radiation and matter eras [79]. As the field approaches the potential minimum, the Universe enters the epoch of cosmic acceleration.

A general problem for supersymmetric quintessence models is that supersymmetry must be broken if it is to be realized at all in nature. In the gravity and gauge mediated scenarios, the supersymmetry breaking is supposed to occur for the energy scale larger than $\langle F \rangle^{1/2} \gtrsim 10^{10}$ GeV and $\langle F \rangle^{1/2} \gtrsim 10^4$ GeV (where F^2 is the first term in Eq. (53), i.e., $F^2 \equiv e^{K/M_{\text{pl}}^2} D_i W (K^{ij*}) (D_j W)^*$), respectively, to lift the masses of supersymmetric scalar particles above 10^2 GeV. In order to give a negligible vacuum energy in Eq. (53) we require that the superpotential takes the form $W \sim \langle F \rangle M_{\text{pl}} \sim m_{3/2} M_{\text{pl}}^2$, where $m_{3/2}$ is the gravitino mass [79]. Then the superpotential $W = \Lambda^{3+\alpha} \varphi^{-\alpha}$ used above gets corrected by the term $m_{3/2} M_{\text{pl}}^2$. This gives rise to the correction of the order of $m_{3/2}^2 M_{\text{pl}}^2$ to the quintessence potential, so that the flatness of the potential required for the late-time cosmic acceleration can be spoiled.

Although this problem looks serious, unconventional supersymmetry breaking models in string theory may overcome this problem. In Ref. [80] it was suggested that we may live in a four-dimensional world with unbroken supersymmetry. In this scenario the mass splitting between the superpartners occurs as a result of the excitations of the system while maintaining a supersymmetric ground state. Then we do not need to worry about the contribution of the supersymmetry breaking terms to the quintessence potential.

There are also some supergravity models in which the above mentioned problem can be avoided. In the framework of extended supergravity models [81, 82] the mass squared of any light scalar fields can be quantized in unit of squared of the Hubble constant H_0 of de Sitter solutions. The de Sitter solutions correspond to the extrema of an effective potential $V(\phi)$ of a scalar field ϕ . Around the extremum at $\phi = 0$ the field potential is given by $V(\phi) = \Lambda + (1/2)m_\phi^2 \phi^2$ with $\Lambda > 0$. In extended supergravity theories the mass m_ϕ is related to Λ via the relation $m_\phi^2 = n\Lambda/(3M_{\text{pl}}^2)$, where n is an integer. Since $H_0^2 = \Lambda/(3M_{\text{pl}}^2)$ for de Sitter solutions, $m_\phi^2 = nH_0^2$. In the $\mathcal{N} = 2$ and

$\mathcal{N} = 8$ extended supergravity theories we have $n = 6$ and $n = -6$ respectively [82, 81], so that the field potentials are

$$V(\phi) = 3H_0^2 M_{\text{pl}}^2 [1 \pm (\phi/M_{\text{pl}})^2] . \quad (58)$$

The energy scale of the supersymmetry breaking is determined by the constant Λ . If the potential (58) is responsible for dark energy, we require that $\Lambda \approx H_0^2 M_{\text{pl}}^2 \approx 10^{-47} \text{ GeV}^4$. The supersymmetry breaking scale is so small that the ultra light mass of the order of 10^{-33} eV can be protected against quantum corrections.

The PNGB models based on the potential (25) also allow to protect the light mass of quintessence by the $U(1)$ symmetry. An example of a very light PNGB is the so-called axion field, which was originally introduced to address the strong CP problem [83]. When a global $U(1)$ symmetry is spontaneously broken, the axion appears as an angular field ϕ with an expectation value $\langle \varphi \rangle = f_a e^{i\phi/f_a}$ of a complex scalar at a scale f_a . In string theory there are many light axions, possibly populating each decade of mass down to the scale $H_0 \approx 10^{-33} \text{ eV}$ [84]. In the limit $\mu \rightarrow 0$ the potential vanishes, so that the symmetry becomes exact. The radiative corrections to V do not give rise to an explicit symmetry breaking term because they are proportional to μ^4 . Hence the small mass associated with dark energy can be protected against radiative corrections.

If the PNGB potential (25) is responsible for the cosmic acceleration today, we require that $H_0^2 \approx \mu^4/M_{\text{pl}}^2$ and hence $\mu \approx 10^{-3} \text{ eV}$. The field mass squared around $\phi = 0$ can be estimated as $m_\phi^2 \approx -(M_{\text{pl}}^2/f_a^2)H_0^2$. The slow-roll condition, $|M_{\text{pl}}^2 V_{,\phi\phi}/V| \lesssim 1$, translates into $f_a \gtrsim M_{\text{pl}}$. Then the field mass is constrained to be $|m_\phi| \lesssim H_0$, so that the field starts to evolve only recently. As we studied in Sec. 3.3, this belongs to the class of thawing quintessence models.

In supersymmetric theories there have been a number of attempts to explain the small energy scale $\mu \approx 10^{-3} \text{ eV}$ [41, 42, 43, 44]. Hall *et al.* [44] tried to relate μ with two fundamental scales, the Planck scale $M_{\text{pl}} \approx 10^{18} \text{ GeV}$ and the electroweak scale $v \approx 10^3 \text{ GeV}$. There is the induced seesaw scale $v^2/M_{\text{pl}} \approx 10^{-3} \text{ eV}$, which is of the same order of μ . If we assume the relation $\mu \approx v^2/M_{\text{pl}}$ and $f_a = M_{\text{pl}}$, it follows that $|m_\phi^2| \approx \mu^4/f_a^2 \approx v^8/M_{\text{pl}}^6$. This gives rise to the mass of the order $|m_\phi| \approx v^4/M_{\text{pl}}^3 \approx 10^{-33} \text{ eV}$.

In order to justify the relation $\mu \approx v^2/M_{\text{pl}}$, Hall *et al.* [44] proposed supersymmetric models with an axion in a hidden sector. In this set up the axion ϕ has interactions with the quarks q, q^c in the form $\mathcal{L}_{\text{int}} = m_q q q^c e^{i\phi/f_a}$ at a scale M , where m_q is the quark mass of the order of the effective supersymmetry breaking scale $m_B = v^2/M_{\text{pl}}$. If at least one of the quark flavors has a mass smaller than the order of M , a quark condensate forms such that $\langle q q^c \rangle \approx M^3 e^{i\tilde{\phi}/M}$ with an angular field $\tilde{\phi}$. This gives rise to the axion potential $V = m_q M^3 \cos(\phi/f_a + \tilde{\phi}/M)$, where M is close to m_B . Then the scale μ is of the order of $\mu \approx m_B = v^2/M_{\text{pl}}$.

In summary the thawing models based on the potentials (25) and (58) are good candidates of quintessence from the theoretical point of view.

6. Conclusions

We have reviewed theoretical and observational aspects of quintessence. We classified quintessence models in terms of the evolution of the field equation of state w .

In tracking models the solutions with different initial conditions converge to a common trajectory characterized by the analytic solution (21). A typical example of this class is the potential $V(\phi) = M^{4+p}\phi^{-p}$ ($p > 0$), in which case w is nearly constant ($w_{(0)} = -2/(p+2)$) during the matter era. The joint data analysis of SN Ia, CMB, and BAO gives the bound $w_{(0)} < -0.964$ (95 % CL) and hence the deviation from the Λ CDM is small. The inverse power-law potential appears in a fermion condensate model of a globally supersymmetric gauge theory, but the theoretical values of p are larger than those constrained by observations.

The exponential potential $V(\phi) = V_0 e^{-\lambda\phi/M_{\text{Pl}}}$ gives rise to a scaling solution along which $w = w_m$ and $\Omega_\phi = 3(1+w_m)/\lambda^2$. Under the condition $\lambda^2 > 3(1+w_m)$ the scaling solution is an attractor during the radiation and matter eras, but it does not exit to the epoch of cosmic acceleration. This problem can be alleviated for the double exponential potential (22) or for the potential (56) appearing in the context of supergravity. The likelihood analysis for the potential (22) with $\lambda_2 = 0$ shows that the transition from $w = 0$ to $w = -1$ needs to occur at the early cosmological epoch ($a_t < 0.23$ (95 % CL) according to the parametrization (23) with $w_p = 0$ and $w_f = -1$).

In thawing models there is an analytic solution (32) of w written in terms of the three parameters w_0 , $\Omega_{\phi 0}$, and K . The parameter K is related to the mass of quintessence. We require the condition $K \lesssim 10$ to avoid the rapid roll down of the field along the potential. Under the prior $w_0 > -1$, the today's field equation of state is constrained to be $w_0 < -0.695$ (95 % CL) from the joint data analysis of SN Ia, CMB, and BAO. The potential (25) of PNGB and the potentials (58) appearing in extended supergravity theories belong to the class of thawing models. In these models, the small field mass m_ϕ associated with dark energy can be protected against radiative corrections due to underlying symmetries.

In order to confront quintessence models with the observations of redshift-space distortions of clustering pattern of galaxies, we derived analytic formulas for the growth rate $f(z)$ as well as $f(z)\sigma_8(z)$ of matter density perturbations. These are useful to place constraints on the quintessence equation of state. We expect that future high-precision observations of RSD, combined with other measurements such as SN Ia, CMB, BAO, and weak lensing, will allow us to distinguish quintessence from Λ CDM.

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